university of groningen

Solutions - Resit (Group A) January 28 2021

1. Let S_{13} be the permutation group on $\{1,2,3,\dots,13\}$ and let $\sigma = (3\ 4\ 7\ 8\ 2)(1\ 3\ 8\ 4)(1\ 2\ 4\ 7)(6\ 9) \in S_{13}$.

   (a) [**1 point**] Write σ as a product of 2-cycles.

   Solution: One can write
   $$\sigma = (3\ 4)(4\ 7)(7\ 8)(8\ 2)(1\ 3)(3\ 8)(8\ 4)(1\ 2)(2\ 4)(4\ 7)(6\ 9).$$
   Any correct answer is 1 point.

   (b) [**2 points**] Write the inverse of σ as a product of disjoint cycles.

   Solution: We write this permutation as a product of disjoint cycles:
   $$\sigma = (1\ 3\ 2)(4\ 8\ 7)(6\ 9).$$
   Since disjoint cycles commute the inverse of σ is the product of the inverses of the cycles:
   $$\sigma^{-1} = (1\ 2\ 3)(4\ 7\ 8)(6\ 9).$$
   Writing as a product of disjoint cycles is 1 point; finding the inverse is 1 point.

   (c) [**2 points**] Compute the kernel of the homomorphism
   $$\varphi : \mathbb{Z} \to S_{13}$$
   $$k \mapsto \sigma^k.$$

   Solution: By definition
   $$\ker(\varphi) = \{k \in \mathbb{Z} \ : \ \sigma^k = (1)\}.$$
   This implies that $\mathrm{ord}(\sigma) = \mathrm{lcm}(3,3,2) = 6$ divides k. Therefore we have $\ker(\varphi) = 6\mathbb{Z}$.
   Arguing that $\mathrm{ord}(\sigma) \mid k$ is 0.5 points, computing $\mathrm{lcm}(3,3,2) = 6$ 0.5 points and concluding $\ker(\varphi) = 6\mathbb{Z}$ is 1 point.

   (d) [**2 points**] Find a subgroup of S_{13} containing σ and isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

   Solution: By CRT, we have $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}/30\mathbb{Z}$. So we need to find a cyclic subgroup of order 30 which contains σ. The cyclic group generated by
   $$\tau = (1\ 3\ 2)(4\ 8\ 7)(6\ 9)(5\ 10\ 11\ 12\ 13)$$
   contains σ since $\tau^{25} = \sigma$. Moreover, since $\mathrm{lcm}(3,3,2,5) = 30$, this subgroup consists of 30 elements.
   If they figure out that the group should be generated by τ and a 5-cycle then 1 points, and if they choose a correct 5-cycle (i.e., a disjoint one) then it is 2 points. They should also argue why these permutations generate a group isom. to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ (if they don't then deduct 0.5 points).

2. Consider the group

$$G = \left\{ \begin{bmatrix} x & y \\ z & t \end{bmatrix} : x, y, z, t \in \mathbb{Z}/5\mathbb{Z},\ xt - zy \neq 0 \right\}$$

with respect to matrix multiplication. Let

$$H = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{Z}/5\mathbb{Z},\ a \neq 0 \right\}.$$

(Recall that $(\mathbb{Z}/p\mathbb{Z})^{\times}$, where p is a prime, is a group with respect to multiplication.)

(a) [**3 points**] Show that H is a subgroup of G but not a normal subgroup.

Solution: Since $a \neq 0$ we have $a \cdot 1 - 0 \cdot b \neq 0$, so $H \subset G$. [**0.5 points**]

- [**0.5 points**] $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in H$;
- [**0.5 points**] For every $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix} \in G$ we have

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ac & ad + b \\ 0 & 1 \end{bmatrix} \in G.$$

- [**0.5 points**] For every $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in G$ the matrix $\begin{bmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{bmatrix} \in G$ (as $a \neq 0$ and every non zero element in $\mathbb{Z}/5\mathbb{Z}$ is invertible) such that

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So H is a subgroup of G. By definition H is normal iff $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$ so to show that H is not normal it suffices to find $g \in G$ and $h \in H$ such that $ghg^{-1} \notin H$. Let

$$g = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in G \quad \text{and} \quad h = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in H.$$

Then we get

$$ghg^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \notin H.$$

[**1 point**]

(b) [**5 points**] Find the Sylow p-groups in H. Which ones are normal in H?

Solution: Since $a, b \in \mathbb{Z}/5\mathbb{Z}$ and $a \neq 0$ we have $|H| = 20 = 2^2 \cdot 5$. The possible Sylow p-subgroups are Sylow 2-groups of order 4 and Sylow 5-groups of order 5. We first determine n_2 and n_5. By Sylow theory we have

$$n_2 \mid 5 \quad \text{and} \quad n_2 \equiv 1 \mod 2$$

$$n_5 \mid 4 \quad \text{and} \quad n_5 \equiv 1 \mod 5.$$

Therefore we get $n_2 = 1$ or 5 and $n_5 = 1$. So there is only one Sylow 5-group, denote it by P_5. Since $|P_5| = 5$, it is cyclic so we need to find an element in H of order 5. We determine $a, b \in \mathbb{Z}/5\mathbb{Z}$ such that

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^5 = \begin{bmatrix} a^5 & a^4b + a^3b + a^2b + ab + b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So one can take $a = 1$, and $b = 1$. So $P_5 = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$.

We will now determine the Sylow 2-groups. Let P_2 be a Sylow 2-group. Since $|P_2| = 4$, we have

$$P_2 \cong \mathbb{Z}/4\mathbb{Z} \quad \text{or} \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

We have

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^4 = \begin{bmatrix} a^4 & a^3b + a^2b + ab + b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, an element has order 4 iff $a^4 = 1 \mod 5$, $a^2 \neq 1 \mod 5$ and $a^3b + a^2b + ab + b = 0 \mod 5$ iff $(a, b) \in \{(2, 0), (2, 1), (2, 2), (2, 3), (2, 4)\}$. Therefore there are 5 Sylow 2-groups and they are isomorphic to $\mathbb{Z}/4\mathbb{Z}$, which are

$$\left\langle \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

The Sylow 5-group is the only Sylow normal subgroup.
Computing n_2 and n_5 is 1 point; writing the Sylow 5-group is 1 point; showing that the Sylow 2-group is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ is 1 point; writing the Sylow 2-groups is 1 point; determining the normal ones correctly is 1 point.

(c) [**3 points**] The group H acts on $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ via the rule

$$\left(h, \begin{bmatrix} x \\ y \end{bmatrix}\right) \mapsto h \begin{bmatrix} x \\ y \end{bmatrix}.$$

Is this action faithful; transitive; or fixed point free? Explain.

Solution:

- [**1 point**] The action is faithful: Let $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix}$ be arbitrary elements in H. Then we have

$$\begin{bmatrix} ax + by \\ y \end{bmatrix} \neq \begin{bmatrix} cx + dy \\ y \end{bmatrix} \iff ax + by \neq cx + dy \mod 5 \iff (a - c)x \neq (d - b)y \mod 5.$$

  If $b \neq d$, one can take $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and if $a \neq c$, one can take $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This proves that the action is faithful.
- [**1 point**] The action is not transitive: take $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then

$$H \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \ : \ a \in \mathbb{Z}/5\mathbb{Z} \right\} \neq \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}.$$

- [**1 point**] The action is not fixed point free since it fixes $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(d) [**2 points**] Write $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ as a union of disjoint orbits of the action in (c).

Solution: We have

$$\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} = H \begin{bmatrix} 0 \\ 0 \end{bmatrix} \cup H \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cup H \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cup H \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cup H \begin{bmatrix} 0 \\ 3 \end{bmatrix} \cup H \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

Complete answer is 2 points and any reasonable attempt with a mistake 1 or 1.5 points depending on the mistake.

3. Let G be a group with precisely one non-trivial proper normal subgroup N. Suppose $|N| = 5$.

(a) **[2 points]** If $f : G \to \mathbb{Z}/12\mathbb{Z}$ is a surjective homomorphism, what are the possible orders for the group G?

Solution: By the homomorphism theorem, we have

$$G/\ker(f) \cong \mathbb{Z}/12\mathbb{Z}$$

Since $\ker(f)$ is a normal subgroup in G, we have $\ker(f) = \{1\}, N$ or G. **[0.5 points]** Suppose $\ker(f) = \{1\}$. Then by the homomorphism theorem we have $|G| = 12$ however on the other hand we have $5 \mid |G|$. So we get a contradiction. **[0.5 points]** Since f is a surjective map, the kernel cannot be G. **[0.5 points]** Therefore, we have $\ker(f) = N$ and hence we get $|G|/|N| = 12$ by the homomorphism theorem. So $|G| = 60$. **[0.5 points]**

(b) **[2 points]** Show that there is no injective homomorphism from G to $\mathbb{Z}/12\mathbb{Z}$.

Solution: If there is an injective homomorphism, then the kernel is trivial. Hence the $|G| \mid 12$. This is impossible since $5 \| G|$.
Correct answer is 2 points.

4. (a) **[4 points]** List all non-isomorphic abelian groups of order between 30 and 65 (both bounds are included) with at least two elementary divisors.

Solution: We first list all abelian non-isomorphic groups with two elementary divisors, in other words, we will list abelian groups in the form $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/ab\mathbb{Z}$. These abelian groups have order a^2b for positive integers a, b. **[0.5 points]** So we list $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ with $30 \le a^2b \le 65$:

$$\{(2, b) : 8 \le b \le 16\}$$
$$\{(3, b) : 4 \le b \le 7\}$$
$$\{(4, b) : 2 \le b \le 4\}$$
$$\{(5, 2)\}$$
$$\{(a, 1) : 6 \le a \le 8\}.$$

Complete list is 2 points, deduct points (proportionally) if the list is not complete.
Now assume that there are 3 elementary divisors, i.e, abelian groups are isomorphic to $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/ab\mathbb{Z} \times \mathbb{Z}/abc\mathbb{Z}$ so the cardinality of such groups are in the form a^3b^2c for some positive integers a, b, c.**[0.5 points]** Then we get

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$$
$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}.$$

Same here, each group gives 0.25 points. (0.5 points in total)
If the number of divisors is larger than 3 then there are at least 4 positive integers a, b, c, d. Similarly, we get $30 \le a^4b^3c^2d \le 65$. When $b = c = d = 1$, we get the case $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/a\mathbb{Z}$ with $30 \le a^4 \le 65$. However, this is not possible for any integer a. If any of b, c, d is greater than 1 we are again outside the bounds. So we conclude that the list above is the complete list. **[0.5 points]**

(b) **[3 points]** Find the elementary divisors of the multiplicative group $(\mathbb{Z}/99\mathbb{Z})^\times \times (\mathbb{Z}/56\mathbb{Z})^\times$.

Solution: By the CRT (Theorem II.3.4.) we have

$$(\mathbb{Z}/99\mathbb{Z})^\times \times (\mathbb{Z}/56\mathbb{Z})^\times \cong (\mathbb{Z}/7\mathbb{Z})^\times \times (\mathbb{Z}/8\mathbb{Z})^\times \times (\mathbb{Z}/9\mathbb{Z})^\times \times (\mathbb{Z}/11\mathbb{Z})^\times.$$

[**0.5 points**]
We have $(\mathbb{Z}/8\mathbb{Z})^\times = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$ and since every non identity element in this abelian group has order 2 we have

$$(\mathbb{Z}/8\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

[**0.5 points**]
We have $(\mathbb{Z}/9\mathbb{Z})^\times = \{\bar{1}, \bar{2}, \bar{4}, \bar{5}, \bar{7}, \bar{8}\}$ and since the cardinality of this abelian group is 6 we have

$$(\mathbb{Z}/9\mathbb{Z})^\times \cong \mathbb{Z}/6\mathbb{Z}.$$

[**0.5 points**]
The cardinality of $(\mathbb{Z}/11\mathbb{Z})^\times$ is 10 and since it is an abelian group, it is isomorphic to $\mathbb{Z}/10\mathbb{Z}$. Similarly, the group $(\mathbb{Z}/7\mathbb{Z})^\times$ is isomorphic to $\mathbb{Z}/6\mathbb{Z}$. [**0.5 points**] So we have

$$\begin{aligned}(\mathbb{Z}/99\mathbb{Z})^\times \times (\mathbb{Z}/56\mathbb{Z})^\times &\cong (\mathbb{Z}/7\mathbb{Z})^\times \times (\mathbb{Z}/8\mathbb{Z})^\times \times (\mathbb{Z}/9\mathbb{Z})^\times \times (\mathbb{Z}/11\mathbb{Z})^\times \\ &\cong \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \\ &\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \\ &\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/30\mathbb{Z}.\end{aligned}$$

So the elementary divisors are $2, 2, 2, 6, 30$. [**1 point**]
If they immediately use $(\mathbb{Z}/p\mathbb{Z})^\times \cong \mathbb{Z}/(p-1)\mathbb{Z}$ then don't deduct any points. They should show the other two cases.

5. **True/False.** Prove the following statements if they are correct and disprove if they are wrong.

(a) [**2 points**] The group $G = S_5 \times D_3$ contains a normal subgroup H such that G/H is isomorphic to $\mathbb{Z}/7\mathbb{Z}$.

Solution: **False.** Assume that there is a normal subgroup H in G. Since $|G| = 2^5 \cdot 3^2 \cdot 5$, we have

$$7 = |G/H| = \frac{|G|}{|H|} \in \{k : k \mid 2^5 \cdot 3^2 \cdot 5\}.$$

Since 7 does not divide $|G|$, there cannot be a normal subgroup H such that G/H is isomorphic to $\mathbb{Z}/7\mathbb{Z}$.
Correct answer is 2 points

(b) [**2 points**] $D_9 \times \mathbb{Z}/2\mathbb{Z}$ is isomorphic $D_6 \times \mathbb{Z}/3\mathbb{Z}$.

Solution: **False.** The dihedral group D_9 has an element of order 9, namely the rotation ρ_9 by angle $2\pi/9$. Therefore, $(\rho_9, 0) \in D_9 \times \mathbb{Z}/2\mathbb{Z}$ has order 9. On the other hand the order of the elements (a, b) in $D_6 \times \mathbb{Z}/3\mathbb{Z}$ is $\text{lcm}(|a|, |b|) = \{1, 2, 3, 6\}$ so there is no element of order 9. Therefore these two groups cannot be isomorphic.
Correct answer is 2 points

(c) [**2 points**] Let G be a group of order 38 and let H, K be subgroups of G such that $H \subset K \subset G$ with $H \neq K$ and $K \neq G$. Then H is the trivial subgroup.

Solution: **True.** By Lagrange's theorem, we have $|H| \mid |K| \mid |G|$ and since $|K| \neq |G|$, we have $|K| \in \{1, 2, 19\}$ but since 2 and 19 are primes and $|H| \neq |K|$, we have $|H| = 1$. So it follows.
Correct answer is 2 points

GOOD LUCK! ☺